OPTIMAL PHOSPHORUS RESPONSE IN SITE-SPECIFIC FARMING[1]

Terry L. Kastens, John P. Schmidt, and Kevin C. Dhuyvetter[2]

Abstract

Site-specific farming has provided an opportunity to collect large amounts of field data, but traditional agronomic yield response models have not been developed to exploit this information. This research presents a yield model that incorporates detailed site-specific field information (e.g., soil pH, soil test P, K, and N, fertilizer rates), providing improved fertilizer decision making. A quadratic yield response function and a modified asymptotic Mitscherlich function are compared for irrigated corn yield response to fertilizer N and P. Both functions fit the data well but result in largely different fertilizer recommendations. To improve fertilizer recommendation decisions, multi-variable yield response functions must be consistent with agronomic theory. A modified Mitscherlich function was estimated using farm-level data for wheat in NW Kansas. Wheat yield appeared to respond to soil test P but not to fertilizer P. After establishing a framework within which fertilizer P can build up soil test P, a simulation was conducted on uniform versus site-specific fertilizer decisions. The total discounted value of fertilizer decisions on 7 future crops over 9 years was \$4 acre^{-1} to \$8 acre^{-1} higher for site-specific based decisions than for uniform-based decisions.

Introduction

Traditionally, farm-level fertilizer recommendations depend heavily upon crop response experiments in which spatial variability is severely minimized. Crop yield is often specified as a mathematical function of fertilizer quantities in order to interpolate and extrapolate from measured fertilizer quantity / crop yield relationships. Mathematical relationships, referred to as *response* or *production* functions, have typically been simple functions, with only one and sometimes two explanatory variables. However, with the advent of site-specific farming there is an increased interest in understanding broader yield relationships. For example, site-specific fertilizer recommendations might be viewed as dependent on numerous factors other than soil fertility — factors whose impact directly on crop yield or indirectly on the fertilizer - yield relationship must be understood in order to better exploit this information. Furthermore, increased use of farm-level crop response data will mean less homogeneous experimental units and increased demand for information about factors other than the one(s) being measured and their impacts on crop yield.

This paper examines a technique of using empirically measured crop yield and soil

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[2] Kastens and Dhuyvetter are assistant and associate professors, respectively, in the Agricultural Economics Department, and Schmidt is assistant professor in the Agronomy Department, all at Kansas State University.

fertility data to generate response functions, referred to here as yield modeling.[3] The general objective is to improve the ability of agronomic practitioners to exploit farm-level information for fertilizer decision models. More specifically, we especially focus on phosphorus, constructing response functions from two Kansas data sets. We will discuss empirical (farm-level or controlled experiment) relationships in the context of agronomic, mathematical, and economic theory. This work should be considered exploratory, not definitive, because we are just beginning to use these types of yield response functions for developing fertilizer recommendations.

Background for Response Functions

Crop yield observed at location i at time t, depicted as y_{it}, might be considered to be a general function of various explanatory variables, for example x and z, also observed at location i and time t:

$$(1) \quad y_{it} = f(x_{it}, z_{it}) .$$

Practical use of this model to improve fertilizer decisions, however, requires additional information about equation 1. Specifically, what are f, x, and z? Two modeling decisions must be made. First, which explanatory variables should be used in the model, and second, what is the mathematical relationship between x and z and crop yield y? Explanatory variables that have traditionally been considered, because they impact crop yield, include soil chemical properties (e.g., soil test N (STN), soil test P (STP), and soil pH), soil texture, soil organic matter, soil bulk density, etc. These same variables are appropriate for site-specific decision making with the caveat that the data be available at a suitable cost.

In evaluating the most appropriate functional form for equation 1, two criteria should be considered. The functional form must have been reasonably accurate at predicting crop yields, and the form must be consistent with agronomic and economic theory. Capturing important underlying forces with selection of functional form is especially important to ensure a model's reliability into the future (Kastens and Brester, 1996). Depending only on historical curve fitting (selecting a high-ordered functional form that accurately predicts the past) does not typically result in a reliable yield model. A much simpler functional form, one that may do only a mediocre job at predicting the past but captures some reliable underlying causal force, is often desirable. Regardless, consistency with current theories is always critical in generating reliable models of reality.

Once equation 1 is mathematically specified we can test how well it performs using historical data and ask questions regarding consistency with agronomic and/or economic theory. Consider this simple specification of equation 1, in which only two variables, N fertilizer (*nf* or fertN) and P fertilizer (*pf* or fertP) are considered:

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[3] Yield modeling should not be confused with agronomy's crop growth modeling, which often depends on many mathematical equations to simulate different phases of plant growth.

(2) $y_{it} = A + B_1 nf_{it} + B_2 pf_{it}$.

In equation 2 and other equations in this paper, upper-case letters represent constants and lower-case italicized letters or words represent variables. For example, if A, B_1, and B_2, are 40, 0.7, and 1.0, respectively, in a corn yield model, then equation 2 becomes:

(3) $y_{it} = 40 + 0.7\, nf_{it} + 1.0\, pf_{it}$.

Using (3), 80 lbs of fertN and 30 pounds of fertP would be expected to produce 126 bushels per acre — because 126 = 40 + 0.7 * 80 + 1 * 30.

How are the parameters (the upper-case constants) of equation 2 selected? Conceptually, they might come from known agronomic laws governing crop production. In practice, they are typically estimated using actual observations of crop yields associated with different levels of fertN and fertP. More specifically, the following equation is estimated (A, B_1, and B_2 are assigned numerical values) by minimizing the sum of squared errors (Σe_{it}^2).[4]

(4) $y_{it} = A + B_1 nf_{it} + B_2 pf_{it} + e_{it}$.

Once the parameters in (4) are estimated, the errors can be computed, squared, and averaged to provide a measure of in-sample predictive accuracy, mean squared error (MSE). The RMSE (root mean squared error, or $MSE^{1/2}$) can be directly compared with the standard deviation of crop yield (y_{it}). If this standard deviation is close to the RMSE computed in an estimated yield model, the model would likely not predict yield any more accurately than merely using average yield as an estimate of expected yield everywhere.

An estimated yield model can be used to predict yield out-of-sample, which means predicting yield for explanatory variable values that were not actually observed in the original data set (the one used to estimate the parameters). If there were only four different fertN levels in the data set that was used to derive the parameters in (3), for example 0, 30, 60, and 90 lbs of N, then examining the model's yield prediction (assuming fertP is held fixed at some value) at 40 N is an out-of-sample interpolation. And, predicting yield at 100 N is an out-of-sample extrapolation (because 100 is outside of the range of observed fertN values).

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[4] Using some observed data set to choose the parameter values in (4) such that they minimize the model RMSE (same parameter estimates as minimizing the sum of squared errors or the MSE) can be accomplished using solver routines in computer spreadsheets (in Excel® or Lotus 123®, for example). The computer systematically tries alternative parameter values until it finds the ones that minimize RMSE. For many mathematical functions, such as that depicted by (4), the parameters are not directly or indirectly multiplied by other parameters. For such linear-in-parameters functions, finding the parameters that minimize RMSE is merely an analytical formula, rather than some iterative computer optimization process. That analytical formula is used in ordinary least squares regression (OLS), a technique also readily available in computer spreadsheets or other numerical analysis software packages. Either way, the goal is the same: find the parameter values which minimize in-sample RMSE.

Two classes of questions are especially relevant to help determine if an estimated yield model is robust (will have comparable predictive accuracies both in- and out-of-sample). First, do the parameter estimates make sense agronomically, or are they too small or too large? Second, do certain model implications seem unreasonable even if parameter magnitudes appear reasonable? Consider equation 2 or 3. The change in yield for a 1 unit change in N (mathematically, the first-order partial derivative of y with respect to nf, or $\partial y/\partial nf$) is always expected to be B_1 (i.e., 0.7). This might be a reasonable approximation for an interpolation but unlikely to hold true for extrapolations. Specifically, response to fertilizer likely diminishes at some point. Moreover, how would economically optimal fertN rates be determined? Conceptually, fertilizer should be added until the last unit returns just enough added value (through increased crop yield) to cover its cost. However, the model in (2) and (3) depicts a constant fertilizer response regardless of the fertilizer level, implying that, depending on fertilizer and crop prices, optimal fertilizer rates are either 0 or infinity. To find optimal fertilizer rates, derivatives of yield with respect to the fertilizer variables must diminish with increased fertilizer.

Consider a quadratic version of equation 4 while ignoring the e_{it} term:

$$(5) \quad y_{it} = A + B_1 nf_{it} + B_2 pf_{it} + B_3 nf_{it}^2 + B_4 pf_{it}^2 .$$

Now, the marginal change in yield expected, due to a 1-unit change in N, is $\partial y/\partial nf = B_1 + 2B_3 * nf$. As long as $B_1>0$ and $B_3<0$, equation 5 will project diminishing returns to N. That is, yield will rise with increased N, but at a diminishing rate. Since either of the two first derivatives of (5) can be analytically computed, we can determine, for example, the fertP level that maximizes yield by setting $\partial y/\partial pf$ to 0 and analytically solving for pf:

$$(6) \quad pf^{*} = -\frac{B_2}{2B_4} .$$

Profit maximizing fertP levels are determined by setting the first derivative equal to the input-output price ratio, R, which is the price of a pound of fertP divided by the price of a bushel of crop, here corn (R = 0.12 if fertP is \$0.30 lb^{-1} and corn is \$2.50 bu^{-1}). Now, equation 6 is modified to provide the profit-maximizing level of fertP, pf^{**}:

$$(7) \quad pf^{**} = \frac{R - B_2}{2B_4} .$$

Maximum and economically optimal (profit maximizing) yields can be determined by solving equation 5 with nf and pf values derived in equations 6 and 7.

A potential problem inherent to equation 5 is that N and P responses are independent of each other: $\partial y/\partial nf$ is not impacted by pf and vice versa. This can be remedied by including an interaction term, which is based on a variable created by multiplying nf times pf:

$$(8) \quad y_{it} = A + B_1 nf_{it} + B_2 pf_{it} + B_3 nf_{it}^2 + B_4 pf_{it}^2 + B_5 nf_{it} * pf_{it} .$$

With equation 8, the first derivative of yield with respect to fertP depends on both fertP and fertN: $\partial y/\partial pf = B_2 + 2B_4 * pf + B_5 * nf$. Typically, B_5 would be expected to be positive so that yield is more responsive to fertP at higher levels of fertN (the fertilizers are viewed as complementary). Yield- and profit-maximizing fertP levels comparable to those calculated in (6) and (7) now depend on fertN values:

$$(9) \quad pf^{*} = \frac{-B_2 - B_5 * nf}{2B_4} \text{, and}$$

$$(10) \quad pf^{**} = \frac{R - B_2 - B_5 * nf}{2B_4} .$$

Assuming $B_2>0$, $B_4<0$, and $B_5>0$, then yield and profit maximizing levels of fertP will be higher when fertN is increased.

Equations 5-10 illustrate that the quadratic function is an easy functional form with which to work in extracting measures of interest. One feature of the quadratic function, however, is undesirable. As fertP and fertN levels in (5) are increased beyond pf^{*} and nf^{*}, model-predicted yields turn downward and this may not be appealing in a limiting factor framework.

The limiting factor idea is based on Liebig's Law of the Minimum (1855) which states: "The crops on a field diminish or increase in exact proportion to the diminution or increase of the mineral substances conveyed to it in manure. . . by the deficiency or absence of one necessary constituent, all the others being present, the soil is rendered barren for all those crops to the life of which that one constituent in indispensable." This is considered the weak link theory, which indicates that production inputs are generally complementary — increasing levels of one input will lead to yields that plateau, unless other inputs are simultaneously increased, causing the whole plateau to rise. The weak link/yield plateau framework for considering response functions does not allow for yields that are expected to turn downward with increasing input levels.

To see why the quadratic functional form may be somewhat troubling consider figure 1, which shows a hypothetical yield response to a single input. Clearly, the response appears consistent with the yield plateau idea, with yields reaching a maximum of 100 then remaining constant for input rates greater than around 18. Figure 2 shows the results of fitting a quadratic function as well as an asymptotic function (discussed later) to the data. If we accept the quadratic function as a reliable generalization of the underlying causal relationship, management decisions would be different than those implied from figure 1. First, peak yields with the quadratic do not appear to be reached until an input rate of 36, which is double the 18 observed in figure 1. Second, the quadratic function depicts decreasing yields at input rates greater than

36, but this does not seem to represent the data well.[5]

Specific Considerations in Selecting the Functional Form

Although determining the "true" functional form of a given relationship is impossible, consideration of certain functional form characteristics should help determine whether one function might capture agronomic and economic relationships better than other functions, and hence serve as a better predictor of yield when used in an out-of-sample framework. Ultimately, the goal is to select a response function to be used in guiding site-specific fertilizer decisions — in our example, phosphorus recommendations. We present several considerations that we believe are important in determining the "true" functional form of a relationship.

The first consideration is the preference exhibited by agronomists for the plateau-type or asymptotic functions compared to quadratic functions. Recently, a number of agronomic research studies have used such asymptotic functions specifically for phosphorus (Obreza and Rhoads, 1988; Halvorson, 1989; McCollum, 1991; Cox, 1996; Randall, et al., 1997a; Randall, et al., 1997b).

The second consideration is that the functional form should accommodate the weak link or limiting factor idea. One input factor should be allowed to impact the yield response associated with another factor. Additionally, one factor should never be allowed to fully compensate for the lack of another factor (see Bauder et al., 1997 regarding this issue with respect to STP and fertP). More simply, factors should generally be treated as complements rather than substitutes. For example, Halvorsen (1989) reports that ". . . N fertilization significantly improved the recovery of fertilizer P in the harvested grain."

The third consideration is that some factors must be allowed to behave as substitutes. For example, soil test fertility and applied fertilizer are likely to behave as substitutes. Specifically with regard to phosphorus, Sharpley (1986); Pothuluri, et al. (1991); and Bauder et al., (1997) have each found that fertP recovery had an inverse relationship with immediately preceding (in time) residual STP values. Intuitively, applying one additional pound of fertP will increase crop

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[5] Besides those limitations already noted, the quadratic can be difficult to implement in the face of numerous variable interactions. For example, allowing for each interaction, a 7-input model in the form of equation (8) would need 28 parameters for the interaction terms, 7 parameters for the linear variable terms, 7 parameters for the squared terms, and one intercept, for a total of 43 parameters to estimate. Although such a model could be uniquely estimated with as few as 43 observations of data, reliability of estimates would be questionable with that little information. Likely because of the ease of estimation using ordinary least squares regression, as well as the ease of calculating its partial derivatives, the quadratic is still a well-used response function in spite of all of its limitations. Nonetheless, it is important to gain experience with alternative functions precisely because functions other than the quadratic might become increasingly important as response functions are used in site-specific farming. Although the research here focuses on one principal alternative to the quadratic, the interested reader should examine *Selected Functional Forms in Production Function Analysis* (Griffin, et al., 1987). The authors examine 20 different functional forms, considering that functions contain both maintained as well as testable hypotheses.

yield more when STP is low than when it is high.

The fourth consideration is to select a function that does not equate zero yield with a zero level of an input. The fifth consideration is that the function must be able to accommodate "bads," in which excessive input levels are expected to lead to reduced yields. The sixth consideration is that the function must accommodate special variables such as pH, for which maximum yield is not expected to occur at low or high values, but at some pH value in between. These functional form requirements are enumerated in table 1.

The first point (Table 1) eliminates from consideration all but 4 of the 20 functional forms discussed by Griffin, et al. (1987). One of the remaining four forms is eliminated because it does not allow for 0 input values (the function is undefined there). Another functional form is ruled out because it has parameter values directly raised to the power of variable values, which would contribute to substantial data scaling problems. Of the two remaining functions, the Resistance function and the Mitscherlich function, only the Mitscherlich is common in the agronomic literature. Furthermore, the Mitscherlich appears more mathematically straightforward than does the Resistance. Consequently, this research proceeds by selecting the Mitscherlich function as a starting point for an appropriate functional form.[6]

The Mitscherlich Response Function

Using a simple two-variable (N and P) response function as an example, the Mitscherlich specification listed in Griffin, et al. (1987) is:

$$(11) \quad y_{it} = A * \left(1 - e^{-B_1 nf_{it}}\right) * \left(1 - e^{-B_2 pf_{it}}\right). \qquad A, B_1, B_2 > 0$$

Unfortunately, as specified, equation 11 still violates several of the principles listed in table 1. Specifically, principle 3 is violated because (11) only allows for complementary relationships. Principle 4 is violated because 0 input of any factor would imply 0 yield (because $e^0 = 1$). Further, equation 11 does not accommodate principles 5 and 6. With several modifications to the Mitscherlich equation, we can accommodate these principles.

The modification to accommodate principle 4 is relatively easy, although at the expense of an added parameter for each explanatory variable (a G term for each):

$$(12) \quad y_{it} = A * \left(1 - G_1 e^{-B_1 nf_{it}}\right) * \left(1 - G_2 e^{-B_2 pf_{it}}\right). \qquad 0 < G_1, G_2 < 1$$

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[6] A quadratic plateau function, often used in the research reported above around the first point (plateau-type functions), but not considered by Griffin, et al. (1987), was also not considered here. Although it might have appeal in the case of single-variable response functions, it is not immediately clear how multiple variables might be most appropriately incorporated. At the least, in the case of several explanatory variables, the quadratic plateau would be subject to the same "curse of dimension" noted earlier about the quadratic — measuring all interactions of interest involves numerous parameter estimations.

Now, when $nf = 0$, the parenthetical term goes to $1\text{-}G_1$, instead of to 0, which implies expected yields will be >0.

As given in (12), the M-function (a modified Mitscherlich function) treats variables as "goods." Increasing the level of one factor while holding the others constant will increase yield. A variable that needs to be treated as a "bad" must only have that variable's name in (12) replaced with a transformed variable that rises when the original variable falls. For example, if nf_{it} happened to be a bad, it would merely need to be replaced with the term $(K - nf_{it})$, where K is an arbitrary user-selected constant that is chosen to be large enough that any observed or extrapolated nf_{it} of interest will not cause $(K - nf_{it})$ to become negative. This accommodates principle 5 in table 1.[7]

Variables like pH can be treated similarly to the modification needed for a "bad." Namely, the variable term merely needs to be transformed. Consider a variable, x, that is expected to behave like pH. This variable would enter the M-function in this fashion:

$$(13) \quad y_{it} = A * \left(1 - G_1 e^{-B_1 nf_{it}}\right) * \left(1 - G_2 e^{-B_2 pf_{it}}\right) * \left(1 - G_3 e^{-B_3 \{K_3 - [(x_{it} \le D_3) - (x_{it} > D_3)] \cdot (D_3 - x_{it})\}}\right) .$$

The $x_{it} \le D_3$ term and $x_{it} > D_3$ term are Boolean logic terms which assume the value 1 if true and 0 if not true. D_3 is a parameter than can be predetermined (e.g., for pH it could be fixed at 6.5) or one that can be selected like any other parameter — in the squared error minimization framework. The user-selected constant, K_3, must be chosen to be large enough to keep all transformed values positive. Consider an x variable that has the following ordered values: {3,4,5,6,7,8,9}. Suppose K_3 is selected to be 7 and D_3 is selected or happens to be 6 in the parameter estimation process. Then, the x series is transformed into the following (using the curly bracketed specification shown to be multiplied by $-B_3$ in equation 13): {4,5,6,7,6,5,4}. In the transformation, x values of 3 or 9 will both behave the same (as a 4). Similarly, a 4 and 8 will behave the same (as a 5), and so on. This modification (equation 13) accommodates principle 6.

Now, equation 13 must only be modified to accommodate principle 3, which allows some factors to behave as substitutes. All "goods" are automatically substitutes for any "bads" in the M-function. Consider two variables, $x1$, which is a good, and $x2$, which is a bad, and hence enters the M-function as values subtracted from some arbitrary constant. In this case, $\partial y/\partial x1 \ge 0$, $\partial y/\partial x2 \le 0$, and $\partial^2 y/\partial x1 \partial x2 \le 0$. However, this is not the issue. The issue is accommodating two "goods" that behave as substitutes, i.e., being able to have $\partial y/\partial x1 \ge 0$, $\partial y/\partial x2 \ge 0$, and $\partial^2 y/\partial x1 \partial x2 \le 0$. That is, we would like the first partial derivatives for $x1$ and $x2$ to be positive and the cross partial derivatives to be negative. An example of this is fertP and STP. We would expect both variables to positively impact yield, yet they should behave as substitutes. Lower soil-test phosphorus levels should lead to larger phosphorus fertilizer responses.

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[7] Unfortunately, accommodating a "bad" in (12) is not as simple as allowing that B parameter to be negative (making -B positive). To do so would mean the G-including term could be greater than 1, implying a negative parenthetical term, and hence negative yield projections.

To accommodate two "goods" that require a negative cross partial derivative we could create another variable. In the $x1$, $x2$ framework, the variable to be created is $x1x2 = x1 * x2$. Then, the newly created $x1x2$ would need to enter the model as a "bad" (be subtracted from some constant). Unfortunately the process is not completely straightforward. Inclusion of the $x1x2$ variable in the underlying equation adds a new negative term in the first partial derivative equation (to what before was only a single positive term). If the negative term is large enough, it could overwhelm the positive one, causing the first partial to be negative, leading to unsupported and unexpected results — positive input changes would mean negative yield changes. One way to prevent this is within the computer solution routine. That is, as in-sample RMSE is being minimized, the partial derivatives can be calculated at each iteration, checked for expected signs (are they positive?), and a penalty added to the RMSE when the partials have the wrong signs in order to prevent a nonsensical answer.

Response in Kansas Irrigated Corn

The first data set evaluated was compiled by Alan Schlegel, agronomist-in-charge at the Kansas State University Southwest Research-Extension Center, Tribune. Irrigated corn yields were matched to fertN (N) and fertP (P_2O_5) phosphorus fertilizer levels from 1992 through 1998 with 5 replications each year involving 3 fertP levels (0, 40, and 80 lbs acre^{-1}) and 6 N levels (0, 40, 80, 120, 160, and 200 lbs acre^{-1}), for a total of 630 observations. Unfortunately, soil fertility information was absent from this data set, providing only fertilizer and yield information from which to build response functions.

Figure 3 illustrates yearly average corn yield for the 90 plots. That average yield in 1995 is probably a negative outlier should be considered in a fertilizer response model designed to derive recommended fertilizer rates. We used these data to fit two different response functions, a Q (quadratic) and an M-function. The Q model was specified as

$$(14) \qquad y_{it} = A_0 + A_1 yr95_t + B_1 nf_{it} + B_2 pf_{it} + B_3 nf_{it}^2 + B_4 pf_{it}^2 + B_5 nf_{it}{*}pf_{it} + e_{it} \,,$$

where y_{it} is corn yield, $yr95$ is a binary variable valued at 1 if the observation was from 1995 and 0 otherwise, nf is nitrogen fertilizer (lbs N acre^{-1}), pf is phosphorus fertilizer (lbs P_2O_5 acre^{-1}), i and t index plot and year, respectively, uppercase letters denote parameters to be estimated, and e_{it} is a model prediction error. The estimated model (using ordinary least squares regression) was

$$(15) \qquad \begin{aligned} y_{it} = {} & 47.4874 - 53.9399\, yr95_t + 0.7737\, nf_{it} - 0.0026\, pf_{it} \\ & + 1.5380\, nf_{it}^2 - 0.0163\, pf_{it}^2 + 0.0044\, nf_{it}{*}pf_{it} \,, \end{aligned}$$

with $R^2 = 0.84$ and an in-sample RMSE of 19.93. All parameter estimates were highly statistically significant (different from 0).

Figure 4 graphically depicts yield projections by fertN levels across 3 fertP rates (0, 40, 80) from the quadratic function specified in equation 15. Even though unobserved in the data, yield projections for fertN rates above 200 were considered to determine the sensitivity of the model's projections. Two questions immediately stand out in the figure. First, should the predictions of decreased yield with higher fertN rates, especially obvious in the 0 fertP line, be ignored — especially in light of figure 2, which had shown the quadratic function's strong tendency to turn downward even when the data do not? Second, would 40 lbs of fertP actually be expected to yield more than 80 lbs of fertP at fertN levels less than 100?

Because no information other than fertilizer and yields was available in the corn plot data, specifying an M-function was easy relative to the earlier discussion of the many modifications that might be needed to ensure a reasonable model. The M-model was specified as

$$(16)\qquad y_{it} = A_0 * \left(1 - A_1 yr95_t\right) * \left(1 - G_1 e^{-B_1 nf_{it}}\right) * \left(1 - G_2 e^{-B_2 pf_{it}}\right) + e_{it}\ ,$$

where the notation is the same as described for equation 14. Unlike (14), where the parameters could be estimated analytically using ordinary least squares, the parameters in (16) are highly nonlinear and must be estimated numerically through an iterative process seeking to minimize the sum of the squared in-sample prediction errors (SSE).[8] The estimated M-model is

$$(17)\qquad y_{it} = 193.4046 * \left(1 - 0.4210\, yr95_t\right) * \left(1 - 0.6541\, e^{-0.0124 nf_{it}}\right) * \left(1 - 0.4178\, e^{-0.1047 pf_{it}}\right),$$

with $R^2 = 0.85$ and an in-sample RMSE of 18.99. In spite of estimating only 6 parameters rather than the 7 used in the Q-function, the fit (in-sample prediction accuracy) was still better with the M-function than with the Q-function (R^2 was higher and RMSE lower).

The first term in (17) shows that maximum corn yield expected in this framework is 193.4 bu acre^{-1}. The next term shows that, after adjusting for differences in yield due to fertilizer, yields in 1995 were 42% lower than maximum yield. Figure 5 illustrates model-predicted yield for different levels of fertilizer (setting *yr95* = 0). Figure 5 should be contrasted with figure 4, its quadratic counterpart. One major difference is that the 80 lb fertP line coincides almost identically with the 40 lb fertP line, indicating little to gain by applying more fertP than 40 lbs. At 0 and 40 fertP and high fertN rates (Figure 6) the Q projects much lower yields than does the

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[8] Minimizing SSE is the same as minimizing the mean squared error (MSE) or the root mean squared errors (RMSE). Choosing parameter values to minimize SSE in a problem like (16) can often be handily accomplished in a computer spreadsheet using its solver routine. Each parameter is assigned a separate cell in the spreadsheet, given a "first guess" or starting value, and collectively called the "adjustable cells," which the computer will adjust in its attempt to minimize SSE. The adjustable cells are then incorporated into spreadsheet formulas that mimic the right hand side of (16), ultimately providing a prediction of yield for each observed combination of fertN and fertP. The predicted yields (630 for this data set) are then subtracted from the actual or observed yields to become the errors, which are squared and subsequently summed. The spreadsheet cell with that sum is the one the computer is told to minimize — subject to user-selected constraints. Here, A_1, G_1, and G_2 were restricted to be between 0 and 1, while A_0, B_1, and B_2 constrained to be positive. We used a Lotus 123® spreadsheet to estimate the parameters in (16), which were very close to those estimated using a more sophisticated optimization software package, Matlab®. Unlike with ordinary least squares, with nonlinear optimization the user is never sure that the computed minimum SSE is the true minimum. Thus, to improve the confidence level, the solution process should be completed several times, each time with different starting values for the parameters estimated (the adjustable cells).

M-function. But, at 80 fertP, across all fertN levels, the Q- and M-functions predict similarly (Figure 7).

Agronomically and economically optimal fertilizer rates and yields were calculated from the estimated functions. Table 2 condenses this information and provides model fit diagnostics. Although yield predictions are not largely different across the functions, recommended fertilizer rates certainly are. At 263 lbs, the M-function recommends sharply higher fertN rates than does the quadratic (200 lbs.). But, at 40 lbs fertP, the M-function's fertP recommendation is less than the quadratic's at 72. Even with relatively good model fit in a "clean" data set for irrigated corn, fertilizer recommendations are not straightforward.[9]

Response in Kansas Non-irrigated Wheat Using Farm Level Data

The second data set evaluated was comprised of field-specific measurements from 1994 to 1999 for wheat production on a single farm in northwest Kansas (Rawlins County). Normal annual rainfall for Rawlins County is 21 inches. Soils are predominantly silt loams. The data set included the following number of records by year: 1994-17, 1995-17, 1996-13, 1997-10, 1998-20, 1999-15. In addition to records of wheat yield and fertilizer quantity (lbs fertN and lbs fertP as P_2O_5), records included measures of pre-plant soil fertility (STN, STP as Bray P1 and STK as ppm K), organic matter, soil pH, percent sand, percent clay, and percent silt. As judged by the producer, expected yields without hail were assigned to field-years that were subject to substantial hail (because, in that area, hail typically falls near harvest, when most of the crop's nutrients should have been extracted from the soil). In addition, seven fields in 1995 had a substantial frost in May that caused those records to receive special treatment as a binary variable. To establish some familiarity with the non-irrigated wheat data, summary statistics across the 92 field-year records are presented in table 3.

We would expect that, besides soil fertility information, factors such as organic matter, pH, and soil particle size (%sand, %clay, %silt) might also be important for determining optimal fertilizer rates. Thus the model estimated with this data set was designed to incorporate these factors. In the model, soil pH was treated like the *x* variable in equation 13, where peak yields are expected to occur at pH ($0<pH<\infty$), rather than at ∞, like the typical M-function term, or at 0, like a factor considered to be a "bad." Whether soil particle size variables tend to be yield enhancing or decreasing is not known *a priori*, hence these variables were also treated like pH.

Pothuluri, et al. (1991); Randall, et al. (1997b); and Bauder, et al. (1997) each supports the following statement. Relatively speaking, at low levels of STP, changes in fertP are more important determinants of eventual crop yield than are changes in STP. On the other hand, when STP is high, changes in STP are more important determinants of crop yield than are changes in

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[9] In a personal communication, Alan Schlegel indicated that he was reluctant to recommend fertN levels outside the range of his experiment, suggesting that perhaps the fertN recommendation should be taken from the quadratic. He further indicated that fertP levels of 72 were high relative to typical producer fertP rates in the area, suggesting that perhaps the fertP recommendation should be taken from the M-function.

fertP. This statement, along with the idea that a portion of fertP might be tied up in less-plant-soluble forms (Moore et al., 1957), supports the following idea. Phosphorus fertilizer placed in the soil this year is used in one of three ways: 1) by the plant in making crop production, 2) to build up a phosphorus bank or reserve that potentially can be used in future crop production, or 3) by combining into unusable chemical compounds (at least as long as the soil is not allowed to be "mined out" of phosphorus).

The preceding paragraph points to at least three modeling issues. First, although we expect yield response to fertP and to STP each to be positive, we expect the cross partial derivative to be negative, i.e., $\partial^2 y/\partial pf \partial ps \leq 0$, where y, pf, ps, are yield, P fertilizer, and P soil test respectively. That is, the response to fertP should diminish with increased STP. Second, we might desire to allow the relationship between fertilizer yield response and soil test yield response to differ substantially from low testing soils to high testing soils. Third, we need to be prepared for the eventuality that yield may show substantial response to STP but little response to fertP (as noted by Bauder et al., 1997). In that case we would need some estimation of how fertP is transformed to STP over time — in order to make today's fertilizer decisions in a way to maximize future profits from crop production.

Chemically, we would expect 1 lb of P_2O_5 to equate to around 0.44 lb. of P or to roughly 0.22 ppm in the top 0-6" of soil (assuming the top 6 inches is about 2 million lbs.). This means that, if P_2O_5 converted perfectly to STP, every $1/0.22 = 4.5$ lbs of P_2O_5 applied above crop removal would increase STP by 1 ppm. Of course, fertilizer does not convert perfectly to STP, rather a portion gets tied up in less extractable forms. The actual rate of conversion expected is an empirical issue and depends on soil type and soil fertility among other factors.

After making the necessary numerical transformations to compare research results (we converted one example using Mehlich I to Bray P1 by multiplying it by 1.25), we have observed in the literature a wide array of transformation rates between excess (above crop removal) P_2O_5 fertilizer in lbs $acre^{-1}$ in one year to change in Bray P1 ppm over the next year. Barber (1979), looking at soils from the Purdue Agronomy Farm, reports that "Total P [Bray P1] in the soil changed one µg/g for each 4.5 kg P/ha difference between P removed in cropping and P applied as fertilizer." That transformation rate translates to roughly 9 lbs. of P_2O_5 fertilizer to 1 ppm change in STP. Peck, et al. (1971), in low-testing Illinois soils, noted repeatedly that ". . . 1 year after the application of P fertilizer the P-1 test values were increased by approximately one-fourth the amount of P added." This translates roughly to 18 lbs. P_2O_5 per 1 ppm increase in STP. McCollum's (1991) look at North Carolina soils translated to a transformation rate of 14.5 lbs. P_2O_5 to 1 ppm. Randall, et al. (1997b), in Minnesota glacial till soils, report transformation rates for different experiments and soils, ranging from about 26 to 33. They note that Schulte and Kelling (1991) found the low rate of only 9 lbs. of P_2O_5 to raise STP by 1 ppm in Wisconsin soils. Finally, McCallister et al. (1987) report a transformation rate in Nebraska soils equivalent to 33 lbs. of P_2O_5 for a 1 ppm increase in STP.

How do the preceding transformation rates compare to equivalent measures on the NW Kansas study farm? Using P removal rates of 0.6 lbs of P_2O_5 per bushel of wheat and 0.38 lbs per bushel of corn, phosphorus removal was calculated for 89 field records over 1994-1999 where initial and ending STP were each available. The mean starting and ending STP's were 15.93 and 15.37 ppm, respectively. On average, 10.96 less lbs of P_2O_5 were applied as fertilizer than were removed by crop. Using only these means, the transformation rate is 10.96/(15.93-15.37) = 19.6. On the other hand, the question might be asked, Which constant transformation rate, when used on each of the 89 field records, would have provided the most accurate prediction of change in STP (using RMSE as a criterion)? In that case the answer would be 10.1. Given the wide range reported in the literature and the wide range reported here, it seems reasonable to use a transformation rate of around 15 in research with this data set (average of 19.6 and 10.1).

With the groundwork in place, the yield response function estimated using the data from the NW Kansas farm was of the following M-function form (dropping the *i* and *t* indexing subscripts and the trailing error term to simplify the presentation):[10]

$$
\begin{aligned}
y = {} & A_0 * \left(1 - A_1 frost\right) * \left(1 - G_1 e^{-B_1 nf}\right) * \left(1 - G_2 e^{-[B_{2a}(ps \le 15) + B_{2b}(ps > 15)]pf}\right) \\
& * \left(1 - G_3 e^{-B_3 ns}\right) * \left(1 - G_4 e^{-[B_{4a}(ps \le 15) + B_{4b}(ps > 15)]ps}\right) \\
& * \left(1 - G_5 e^{-B_5 ks}\right) * \left(1 - G_6 e^{-B_6 om}\right) \\
& * \left(1 - G_7 e^{-B_7(15000 - nf \cdot ns)}\right) * \left(1 - G_8 e^{-[B_{8a}(ps \le 15) + B_{8b}(ps > 15)] \cdot (15000 - pf \cdot ps)}\right) \\
& * \left(1 - G_9 e^{-B_9\{70 - [(\%sand \le D_9) - (\%sand > D_9)] \cdot (D_9 - \%sand)\}}\right) \\
& * \left(1 - G_{10} e^{-B_{10}\{70 - [(\%clay \le D_{10}) - (\%clay > D_{10})] \cdot (D_{10} - \%clay)\}}\right) \\
& * \left(1 - G_{11} e^{-B_{11}\{80 - [(\%silt \le D_{11}) - (\%silt > D_{11})] \cdot (D_{11} - \%silt)\}}\right) \\
& * \left(1 - G_{12} e^{-B_{12}\{8.5 - [(ph \le D_{12}) - (ph > D_{12})] \cdot (D_{12} - ph)\}}\right) ,
\end{aligned}
\tag{17}
$$

where y is wheat yield, *frost* is a binary variable valued at 1 if a late spring frost harmed the yield in that field and 0 otherwise, *nf* and *pf* are fertN (lbs $acre^{-1}$) and fertP (lbs P_2O_5 $acre^{-1}$), respectively, *ns* and *ps* are STN (lbs $acre^{-1}$) and STP (Bray P1), respectively, *ks* is soil test K (ppm), *om* is soil test organic matter (%), %sand, %clay, and %silt are corresponding measures of soil particle size, *ph* is soil test pH, and upper-case letters are parameters to be estimated (33 total).

Although the asymptotic "limiting factor" concept is generally embodied in equation 17, the equation is modified to accommodate each of the phosphorus response concerns discussed earlier. Notice in particular that the phosphorus terms (associated with G_2, G_4, and G_8) have unique B parameter estimates depending upon whether STP is below or above 15 (49 of 92

[10] No quadratic function was included for comparison because of the large number of interaction terms that would have been needed. In the top two lines of (17) the 15 in "$ps \le 15$" and "$ps > 15$" arbitrarily splits the STP data range at 15. The constants, 15000, 70, 70, 80, and 8.5, were selected to be sufficiently large that, during estimation or in simulations thereafter, the associated terms would be positive.

observations had STP≤15). That allows the relationship between fertilizer response and soil test response to be different at low and high levels of STP.

Equation 17 was estimated minimizing squared errors, resulting in an RMSE of 8.4174 and an R^2 of 0.4218.[11] Equation 18 presents the estimated equation.

$$
\begin{aligned}
y = {} & 77.766305 * (1 - 0.482748\,frost) * \left(1 - 0.168022\,e^{-0.020423\,nf}\right) \\
& * \left(1 - 0.016943\,e^{-[0.014055(ps \le 15) + 0.000193(ps > 15)]pf}\right) \\
& * \left(1 - 0.000005\,e^{-0.001540\,ns}\right) * \left(1 - 0.999990\,e^{-[0.463938(ps \le 15) + 0.135758(ps > 15)]ps}\right) \\
& * \left(1 - 0.999986\,e^{-0.004978\,ks}\right) * \left(1 - 0.108959\,e^{-0.000035\,om}\right) \\
& * \left(1 - 0.995138\,e^{-0.001370(15000 - nf \cdot ns)}\right) \\
& * \left(1 - 0.272010\,e^{-[0.000049(ps \le 15) + 0.000507(ps > 15)] \cdot (15000 - pf \cdot ps)}\right) \\
& * \left(1 - 0.999996\,e^{-0.170131\{70 - [(\%sand \le 0.001545) - (\%sand > 0.001545)] \cdot (0.001545 - \%sand)\}}\right) \\
& * \left(1 - 0.997894\,e^{-0.143944\{70 - [(\%clay \le 69.998553) - (\%clay > 69.998553)] \cdot (69.998553 - \%clay)\}}\right) \\
& * \left(1 - 0.283077\,e^{-0.055292\{80 - [(\%silt \le 6.958023) - (\%silt > 6.958023)] \cdot (6.958023 - \%silt)\}}\right) \\
& * \left(1 - 0.000092\,e^{-1.997967\{8.5 - [(ph \le 6.832374) - (ph > 6.832374)] \cdot (6.832374 - ph)\}}\right) .
\end{aligned}
\tag{18}
$$

While equation 18 appears to be a substantially detailed equation, the goal was, using agronomic theory, to generate a reasonable response function that could be used to generate fertilizer recommendations for VRT (variable rate technologies) decision making. The model explains 42% of the yield variability in the data ($R^2 = 0.42$), compared to the irrigated corn model's fit of 0.85. Of course, it would be easy to improve the R^2 by not requiring the model to conform as much to theory (e.g., not restrict first partial derivatives to be positive). The assumption is that a model that is more theoretically correct will provide better out-of-sample fit (better able to make applied VRT decisions) in spite of potentially poor in-sample fit.

The leading term in equation 18 shows that the wheat yield maximum (goal?) for this data set is around 78 bu acre^{-1}. That is, if everything measured here is optimal (yield-wise, not profit-wise), wheat yield is expected to be 78 bu acre^{-1}. The second term projects a 48% drop in yields due to late spring frost. The very last term in the equation shows that a pH level of 6.8 is expected to maximize yield.

[11] A genetic algorithm was coded within Matlab® software in order to estimate (17). Genetic algorithms are robust optimization techniques that are mathematically simple but computationally intense. For a good introduction in this area readers should see Dorsey and Mayer (1995). To ensure consistency with agronomic theory (described earlier), and that fertilizers and soil tests behave as substitutes rather than complements, yield responses for nitrogen and phosphorus fertilizer and soil tests were each restricted to be positive during the estimation process.

The idea of limiting factors in crop production is a relatively easy concept to understand but often a hard one to measure or show — in part because every variable impacts every other one. In order to better understand, it is often necessary to hold most factors fixed while observing yield impacts associated with only one or two factors varying. Furthermore, the observed minimum, maximum, and range of some variable indicate the scope over which variable changes might be considered reasonable. Figure 8 shows the model-predicted yield response for each fertilizer-type variable over the entire range of that variable observed in the data (the ranges can be seen in table 3). For example, for fertN, the 0 on the x-axis corresponds to the data minimum (10.01, from table 3) and the 100 corresponds to the data maximum (83.7, also from table 3). Values in between are prorated accordingly (percentages of the data range added to the data minimum). All other variables are held fixed at their means.

Figure 8 shows that two fertility variables have very little impact on yields, STN and fertP (both coincide and are horizontal). That is not to say that nitrogen and phosphorus are unimportant to wheat production but that their impacts show themselves principally through fertN and STP. In fact, as STP varies over its range (7 ppm - 35 ppm) yield changes by more than any of the variables varying over their respective ranges. The kink in the STP response at 30% of the range, which happens to be at an STP level of 15.4 ppm, is due to allowing yield's response to STP to behave differently at low levels of STP than at high levels (recall that the model switched parameters at 15 ppm STP).

Figure 9 shows the impact of non-fertility variables on wheat yield. Somewhat unexpected is that yield appears to diminish in increased sand and silt but increase in the face of rising clay values. Perhaps the increased water holding capacity of more clay type soils is a benefit. Also surprising is that organic matter did not appear to have a significant impact on wheat yield; nor did pH, which is not too surprising since pH is not typically a problem in NW Kansas soils.

Considering that STP seems to be the relevant phosphorus variable and assuming a linkage can be made between fertP and STP the question is, Which variables might help with VRT in phosphorus? Clearly the most important one in figure 8 is STP itself. That is, it would certainly be important to have site-specific information about STP. But, what about the other variables? How important are they? To get at that we might ask, How is the efficiency (turning phosphorus into yield) of STP impacted by changes in other variables? Figure 10 shows those impacts (leaving out the impact of changes in STP itself). It might be argued that the range on the y-axis is not particularly large but that is an economic issue to be addressed shortly. Clearly, it should be more profitable to build up STP 1 ppm in low-silt areas than in high-silt areas, also in high-clay areas than in low-clay areas. The synergism or complementary nature of fertN and STP is also quite clear, with greater P efficiency at high levels of fertN.

One potentially interesting model result is that yield appears to respond to STK in figure 8 and P efficiency is enhanced by STK in figure 10. K levels in NW Kansas are typically rated very high and no yield response is expected. Could it be that what the model is observing as changes in STK is actually a proxy for other changes. For example, the simple linear correlation between STP and STK in our data set is 0.56. Might that mean that K should be excluded in future renditions of the model? Such questions routinely arise and must be considered in assessing response function results.

Profit Maximizing Phosphorus Fertilizer Decisions

In the irrigated corn model discussed earlier, optimal fertilizer rates were easy to determine because nothing but fertilizer and relative prices impacted that decision. But, that model offered no help for making site-specific decisions — the same level of fertilizers would be recommended for all parts of the field. In the wheat model just discussed, optimal fertilizer rates depend on measures of all other variables in the model, potentially complicating the decision process. Furthermore, as noted, fertP had only marginal yield impacts while STP likely had economically important yield impacts. This means that assumptions about turning fertP into STP are required to operationalize the process.

Conceptually, if it pays to apply fertP today to enhance next year's and subsequent year's production, then agronomically there is little reason to delay putting on the full amount needed to "build the soil up" to the optimal STP level. That is, suppose that today's STP level is 15 ppm Bray P1 and the long run economically optimal STP level is 25 ppm. Then, if it takes 15 lbs of additional (above crop removal) P_2O_5 to increase STP by 1 ppm, 150 lbs. of P_2O_5 should be applied this year — so that, beginning next year and following with every year thereafter, one can reap the benefits of the higher levels of STP. In this setting, the optimal fertP is reduced to a simple future discounting problem: the increased profit in each future year due to increased yield due to increased STP is discounted back to the present (because future dollars are worth less than today's dollars) and maximized by selecting fertP rates this year.

Unfortunately, a number of factors make the fertP decision more complicated than the simplistic "build STP to the long run optimum this year." As already discussed, STP is a dynamic variable, with its levels changing over time based on previous years' STP levels, crop yields, and fertP rates. Further, some fields or areas may have STP levels that are above the economically optimal levels. For those areas it would be more profitable to apply zero fertP — at least until STP levels declined. Consequently, fertP determination is a complex problem, with the decision maker conceptually choosing all future levels of fertP simultaneously.

Because current fertP rates help determine future STP levels and thus future yields, there are a number of reasons that a farmer may choose to apply less than the profit-maximizing quantity of fertP in any particular year. A land rental arrangement might abruptly be ended — before all rewards to fertility improvement can be reaped. A heavy current investment in fertP might constrain investments in other, perhaps more profitable farming decisions. The planned cropping system might change over time, changing the expected P-to-yield relationship. The price of fertP might be abnormally high today. The model on which the fertP recommendation is based might not be quite right and next year's model might generate different recommendations. Thus, even though the agronomic transformation of fertP to future STP might be difficult to explain, the actual decision making process might be even more difficult. In order to complete our economic analysis, we make some simplifying assumptions around the farming practices of our study farm.

1. Cropping pattern and financial discounting

The study farm's cropping pattern is fallow-wheat-corn. Expenditures for corn fertilizer are made in the same year the corn is harvested. However, expenditures for wheat fertilizer are made in the fallow year (ahead of fall planting) but wheat crop sales are not made until the harvest year. Thus, wheat crop sales need to be discounted one year within the wheat crop analysis. Then, gross margins (crop sales less fertilizer expenditures) for future crops (corn or wheat) need to be discounted back to the current decision making period. Assuming an interest or discount rate of i, the appropriate discount factor is d, where $d = 1/(1+i)$. If we are only interested in the upcoming wheat crop, at fertilizing time (say 1999) we would choose the fertilizer amount expected to maximize $\{d * EWS_{2000} - WFC_{1999}\}$, where EWS_{2000} and WFC_{1999} are expected wheat sales in 2000 and wheat fertilizer costs in 1999, respectively. If today's fertilizer decision also impacted the corn crop in 2001 and the following wheat crop harvested in 2003, we would want to choose the current wheat and future corn and wheat fertilizer amounts which maximize $\{d * EWS_{2000} - WFC_{1999}\} + d^2\{ECS_{2001} - CFC_{2001}\} + d^3\{d * EWS_{2003} - WFC_{2002}\}$, where ECS and CFC are expected corn sales and corn fertilizer costs, respectively.

Although the cropping system used on the study farm is fallow-wheat-corn, a suitable corn response function for the farm had not yet been generated at the time of this research. In a study such as this, which depends heavily on discounted future returns, it would be inappropriate to ignore one half of all future crops (the corn part). Consequently, to maintain our framework of two crops in three years, we use wheat results (fertilizer response, crop sales, fertilizer purchases) as a proxy for the corn crop. Thus, in our analysis of a 4-future-crop framework (besides the current wheat crop being planned for) we would choose all current and future fertilizer rates which maximize $\{d * EWS_{2000} - WFC_{1999}\} + d^2\{d * EWS_{2002} - WFC_{2001}\} + d^3\{d * EWS_{2003} - WFC_{2002}\} + d^5\{d * EWS_{2005} - WFC_{2004}\} + d^6\{d * EWS_{2006} - WFC_{2005}\}$.

For this study we assume an interest rate of 9%, or 0.09, leading to a discount factor of d=0.9174. We assume the farmer's acceptance of financial risks associated with buildup of STP is captured by the time horizon he is willing to consider (e.g., an extremely risk averse producer would need to observe an immediate payback to fertilizer). Here, we consider a time horizon of 9 years, encompassing 3 future corn crops (actually, their wheat proxies) and 3 future wheat crops after the wheat crop currently being planned for (a total of 7 crops). Thus, our simulations begin in the fallow year, when fertilizer is to be applied for the wheat crop that will be harvested the following year. To examine the sensitivity of our analyses to the time horizon we also consider a time horizon of 6 years, which allows for 2 future corn crops and 2 future wheat crops after the wheat crop currently being planned for (a total of 5 crops).

2. Fertilizer P and STP buildup

FertN and fertP, applied to wheat ahead of planting, affect the current wheat crop's yield in the manner specified in equation 18 — as do all current measures of other variables. Phosphorus removal, as P_2O_5, for the current wheat crop is considered to be 0.6 lbs bu^{-1} of wheat. P_2O_5 fertilizer amounts above or below removal will build up or decrease STP for the following year at the rate of 15 lbs of excess P_2O_5 equals 1 ppm of STP.

3. Prices

FertN is assumed to cost \$0.15 lb^{-1}, fertP (P_2O_5) \$0.30 lb^{-1}, and wheat is valued at \$3.30 bu^{-1} (corn prices are irrelevant because wheat is used as a proxy for corn).

4. Data variability

This study's response function was estimated using field-level data observations. However, the estimated model (equation 18) can be used to examine fertilizer management at different scales as well, for example site-specific grids. To gain some understanding of the returns to management and technology associated with using a response function to determine fertilizer rates, we consider three different simulations.

The first simulation, referred to as *farm average fertilizer*, merely assigns historical farm average fertilizer rates (53 lb N $acre^{-1}$ and 21 lb P_2O_5 $acre^{-1}$; see table 3) to every field and for each of the 7 projected crops. Model-simulated results (yields, profits, etc.) establish a status quo baseline. The second simulation, *optimal uniform*, chooses fertilizer rates for each future crop that maximize the discounted expected gross margin (wheat sales less fertilizer cost) for all 7 future crops, using data set mean soil fertility measures. The same profit-maximizing rates of N and P fertilizer (14 values — one N and one P level for each of 7 future crops) are then assigned to each observation in the data set and model-simulated results summarized. The third simulation, *optimal VRT*, chooses the fertilizer rates for each future crop which maximize the discounted expected gross margin for all 7 future crops, using soil fertility measures unique to each observation in the data set (each observation in the data set has 14 optimally selected fertilizer values — one N and one P level for each of 7 future crops). The difference in the discounted gross margins for *farm average fertilizer* and *optimal uniform* is an indication of the expected returns to using the response function and related agronomic assumptions about buildup of STP to help select uniform phosphorus fertilizer rates. The difference between *optimal uniform* and *optimal VRT*, is an indication of the expected returns to adopting site-specific technologies, namely variable rate fertilizer.

For our NW Kansas study farm we completed the preceding simulation for two other data sets besides the Farm data set summarized in table 3. The Field #1 data set is comprised of 59 2.5 acre grids from a single field that was sampled in 1998. The Field #2 data set is comprised of 51 2.5 acre grids from a field sampled in 1999. Both fields, however, used the same response function as the Farm — the one estimated using the Farm data set.

5. Profit maximization process

For the simulations involving optimal fertilizer rates (*optimal uniform* and *optimal VRT*), we simultaneously selected all future fertN and fertP rates such that the 7-crop discounted gross margin was maximized. In all simulated optimal fertilizer selections, the value of the late spring frost variable was set to 0 (in order to plan for the "good" years when there is no late spring frost).

Simulation Results

The simulation results are presented in table 4. Panels 1a and 1b show information about the Farm data set, 2a and 2b cover Field #1, and 3a and 3b cover Field #2. N and P fertilizer rates each went up substantially in going from the baseline (*farm avg fertilizer*) to *optimal uniform* in Panel 1a.[12] Needless to say, average STP and wheat yield were also higher accordingly. Returns rose by $9.88 (differences in gross margin), suggesting there could be economic gains to using a response function approach such as the one used here to help make fertilizer decisions — even without making large moves towards site-specific technologies.[13] Interestingly, however, fertN, fertP, STP, and yield each drop when going from *uniform* to *VRT*, yet returns rose by $4.71, indicating improved fertilizer placement.

It should be noted that the $9.88 and $4.71 acre^{-1} values just mentioned are not annual values, rather they are 7-crop totals, discounted to the present. In the Farm example just given (Panel 1a, where *VRT* might imply field-level rather than sub-field management), for *VRT* to be profitable over *uniform*, the $4.71 acre^{-1} would have to cover the added one-time cost of completing field-level soil sampling rather than farm composite soil sampling (after the first year, fertilizer decisions could be based on crop removal and fertP rates rather than soil tests). Secondly, it would have to cover the added costs of making the necessary machinery adjustments etc. to apply different fertilizer rates to each field in each year.

For a farm with ten 100-acre fields, the difference in soil sampling costs (at say $15 per sample) for farm-level vs. field-level is only $0.14 acre^{-1} ((10 x 15 - 15)/1000), leaving $4.57 acre^{-1} to cover machinery adjustments for 7 years. However, if the analysis in Panel 1a can be extrapolated to a farm with ten 2.5-acre fields (as in site-specific farming), the difference in soil sampling costs is $5.40 acre^{-1} ((10 x 15 -15)/25), which is inadequately covered by the $4.71 gains to *VRT* over *uniform*. Regardless, the largest gains observed in Panel 1a have to do with recognizing that P should be treated as a temporal dynamic decision, and that fertP rates should probably be higher than they typically have been on this study farm.

Panel 1b of table 4 shows the 5-crop (6-year horizon) counterpart to Panel 1a. Now, both *uniform* and *VRT* fertP rates are substantially lower than the baseline, yet returns are substantially

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[12] Although not explicitly shown, in this research almost all (over 99%) of each economic difference shown in table 4 has to do with P, rather than N, management. Thus, we make no attempt to illustrate economic differences specific to N and P.

[13] It might be argued that the *farm avg fertilizer* baseline is somewhat of a "straw man" in that it would not be relevant for operators who already manage fertilizer by field. Therefore we could be giving too much credit to the response function approach to fertilizer management rather than to public universities, private consultants, and fertilizer companies who may be making field-level fertilizer recommendations. Nonetheless, it is probably unlikely that providers of fertilizer recommendations explicitly consider the time horizon issue as we do in this research. Furthermore, our study farm's operator indicated that, in spite of taking numerous field-level soil samples, his fertilizer management practices probably fall closer to the *farm avg fertilizer* baseline than to the *optimal uniform* simulation — because of the logistics of machinery and labor management, and admittedly, because of not being quantitatively aware of the expected gains to field-level management over farm-level management.

greater. Interestingly, in spite of *VRT* averaging only 5 lb acre^{-1} of fertP, compared to 21 for the baseline, projected yields were higher with *VRT*. This is a tribute to placing fertP in the right place at the right time.

Figure 11 shows the projected crop average fertP rates and associated changes in STP for the simulation behind Panel 1a. On average, over 115 lbs of fertP were suggested in the first year and around 60 lb in the second year, clearly indicative of a fast buildup of STP — which peaks at around 25 ppm ahead of the third crop. However, because the time horizon is not infinite, rather only 7 crops, the optimal plan places no fertP for crops 5 through 7, allowing STP to diminish to around 15 ppm preceding crop 7. Figure 12 shows fertP and STP information for the 5-crop horizon in Panel 1b. The overall average 5 lb acre^{-1} *VRT* fertP rate in Panel 1b is clearly skewed towards the first two years — timed to merely slow the STP decline over ensuing crops. What figure 12, or the 5 lb value in table 4, do not show is that a number of fields received over 80 lb acre^{-1} of fertP in the first year. These fields had the most to gain with large amounts of added fertP. Consequently, it is placing fertP in the right place and in the right time that causes a mere 5 lb of fertP in a *VRT* setting to garner higher yields than 21 lb in the baseline setting. It is also why managing only 5 lbs of fertP (compared to 0 in *uniform*) can provide gains of $7.19 acre^{-1}.

In the Field #1 analysis in Panel 2a, gains to P management are about evenly split between *farm avg* to *uniform* and *uniform* to *VRT*. Here, the $7.71 *VRT* over *uniform* advantage, like the $7.19 advantage in Panel 1b, is sufficient to cover the additional $5.40 soil sampling cost posited earlier. Again, the *VRT* line, with less fertP than the *farm avg* line, has higher yields. As in Panel 1b, in Panel 2b it is shocking that managing only 3.3 lb acre^{-1} of fertP can result in $2.72 gains.

Results for Field #2 (panels 3a and 3b) confirm the results for the Farm and Field #1. Again, unless time horizons are very short, it appears higher fertP rates should generally be considered on this study farm. If the *uniform* over *avg* and *VRT* over *uniform* gains in each panel of table 4 are added together it appears that gains to improved P management might be around $13 to $15 acre^{-1} — before additional costs are factored in.

Conclusion

Following the description of several basic principles for constructing a yield response function, we have specified a number of requirements for response functions to be useful and reliable in aiding site-specific fertilization decisions. These requirements are 1) the function should be based on an asymptotic convergence towards a plateau yield; 2) the "limiting factor" idea should be embodied in the function (factors impact the yield response of other factors and no factor can fully compensate for the lack of another factor — factors are generally complements); 3) a few factors must be allowed to behave as substitutes; 4) a zero-level of some input should not necessarily imply zero yield; 5) the function must be able to accept input "bads," where more input means lower yields; and 6) the function must be able to accept variables like pH, where yield might be expected to reach a maximum within the input range rather than at either endpoint.

Using controlled experiment data for irrigated corn yield response to N and P fertilizer, this study compared the predictive and management implications of two alternative functional forms: a quadratic and a modified Mitscherlich. Both functions fit the data well but resulted in considerably different fertilizer recommendations. The quadratic function is easy to estimate and hence can easily be used in a preliminary examination of the data. But, it may be agronomically problematic and will require many parameters to be estimated in a multi-variable response function. The modified Mitscherlich, on the other hand, is more complicated mathematically and more difficult to estimate. Yet, it typically has less parameters to estimate and is probably more agronomically appealing than is the quadratic.

A modified Mitscherlich yield response function was estimated using farm-level data from a single study farm in northwest Kansas. Besides fertilizer and soil fertility, organic matter, pH, and soil particle size were allowed to impact wheat yield through a yield response model. Yield appeared to respond to soil test P but not to fertilizer P. A framework was established where fertilizer P above crop removal would allow for buildup of soil test P in future years, hence increased wheat yields. That framework allowed for baseline, profit-maximizing uniform, and site-specific fertilizer decisions to be simulated. In the simulations involving the discounted value of fertilizer decisions on 7 future crops over 9 years and 5 future crops over 6 years, a variable rate fertilizer P program had an advantage over a uniform fertilizer P program in the range of \$2 to \$8 $acre^{-1}$. Overall gains to using VRT *and* the response function approach to improving fertilizer P management are expected to be around \$13 to \$15 $acre^{-1}$.

This work represents exploratory, not definitive, work in the area of building decision-aiding yield response functions using farm-level data from a number of input variables. The process appeared reasonable on western Kansas wheat. Although not tested, the yield response function fertilizer-determining framework should be especially fitting where lime is a routine requirement. Likely, the yield response function framework will not become a black box — academic and professional input will constantly be needed to determine whether yield response implications seem reasonable, how to get from a soil applied nutrient (e.g., lime) to a measurement of interest (e.g., pH), and farmer perceptions of risk.

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Table 1. Suggested requirements for response functions estimated from site-specific data and used for variable rate fertilizer recommendations

1. asymptotic plateau-type convergence (yields flatten out over broad levels of high inputs)
2. "limiting factor" framework; factors impact the yield response of other factors; no factor can fully compensate for the lack of another factor; most factors are considered complements
3. a few factors must be allowed to behave as substitutes
4. a 0-level of some input should not necessarily imply 0 yield
5. must be able to deal with input "bads," where increased levels lead to reduced yields
6. must accommodate special variables like pH, where yield peaks mid-range

Table 2. Implications of quadratic and M-function for yield response to nitrogen and phosphorus fertilizer on irrigated corn, Tribune, Kansas 1992-1998

	Yield maximizing		Profit maximizing	
	Quadratic	M-function	Quadratic	M-function
\$/lb. N fertilizer	\$0.00	\$0.00	\$0.15	\$0.15
\$/lb. P_2O_5 fertilizer	\$0.00	\$0.00	\$0.30	\$0.30
\$/bu. corn	\$2.50	\$2.50	\$2.50	\$2.50
optimal N fertilizer	215	infinite	200	263
optimal P_2O_5 fertilizer	76	infinite	72	40
predicted yield	189.4	193.4	188.8	187.4
no. of parameters estimated	7	6		
R^2	0.8395	0.8544		
in-sample RMSE	19.9287	18.9872		

Table 3. Summary statistics for field records from study farm in NW Kansas, 1994-1999 (Farm data set)

	mean	standard dev.	minimum	maximum
wheat yield (bu $acre^{-1}$)	49.11	11.13	22.00	71.24
fertN (lbs N $acre^{-1}$)	52.94	12.63	10.01	83.70
fertP (lbs P_2O_5 $acre^{-1}$)	20.65	8.67	0.00	52.10
pH	6.85	0.45	5.80	7.80
organic matter (%)	1.48	0.28	0.90	2.00
STN (soil test N in lbs $acre^{-1}$)	27.49	10.67	1.22	48.00
STP (soil test P in ppm Bray P1)	16.22	5.75	7.00	35.00
STK (soil test K in ppm)	600.56	63.94	478.00	755.00
% sand	22.02	10.52	5.00	50.00
% clay	28.18	2.78	24.00	35.00
% silt	49.79	10.18	24.00	70.00

Notes: All soil test information based on 0-8" samples

Table 4. Analysis of model simulated N and P fertilizer programs on wheat: farm average, optimal uniform, and optimal VRT; NW Kansas study farm; 7-crop (9-yr) and 5-crop (6-yr) planning horizons

simulation	(fertN) lb N acre^{-1}	(fertP) lb P_2O_5 acre^{-1}	(STP) soil test P Bray P1	wheat yield bu acre^{-1}	discounted gross margin \$ acre^{-1}	advantage over preceding row \$ acre^{-1}
	Mean of model simulations, data set observations by 7 projected crops					
Panel 1a: Farm data set; 92 field observations by 7 projected future crops						
farm avg fertilizer	53.0	21.0	14.3	50.4	$669.10	------
optimal uniform	65.6	30.4	21.2	53.9	$678.98	$9.88
optimal VRT	64.2	26.5	19.6	53.5	$683.69	$4.71
Panel 1b: Farm data set; 92 field observations by 5 projected future crops						
farm avg fertilizer	53.0	21.0	15.0	50.6	$537.48	------
optimal uniform	63.0	0.0	12.2	49.3	$543.95	$6.46
optimal VRT	62.1	5.0	13.3	50.7	$551.14	$7.19
Panel 2a: Field #1 data set; 59 grid (2.5 acre) observations by 7 projected future crops						
farm avg fertilizer	53.0	21.0	17.7	49.8	$659.47	------
optimal uniform	64.1	22.4	21.6	51.8	$666.67	$7.20
optimal VRT	62.5	18.7	20.1	51.7	$674.38	$7.71
Panel 2b: Field #1 data set; 59 grid (2.5 acre) observations by 5 projected future crops						
farm avg fertilizer	53.0	21.0	18.3	50.0	$529.73	------
optimal uniform	62.4	0.0	15.5	49.2	$541.31	$11.58
optimal VRT	61.6	3.3	16.2	49.9	$544.02	$2.72
Panel 3a: Field #2 data set; 51 grid (2.5 acre) observations by 7 projected future crops						
farm avg fertilizer	53.0	21.0	15.1	51.6	$685.94	------
optimal uniform	65.7	28.7	21.4	54.9	$697.39	$11.45
optimal VRT	69.5	28.1	21.0	55.0	$701.02	$3.63
Panel 3b: Field #2 data set; 51 grid (2.5 acre) observations by 5 projected future crops						
farm avg fertilizer	53.0	21.0	15.7	51.8	$551.04	------
optimal uniform	63.6	0.0	12.9	51.0	$562.31	$11.28
optimal VRT	63.2	4.4	14.0	51.8	$564.55	$2.24

Notes: Yields are projected using equation 18 model. After being initialized at measured values, soil test P values for crops 2 through 7 are determined from previous yields and P fertilizer rates. The *farm average fertilizer* simulation summarizes model simulated results after assigning 53 lb N fertilizer acre^{-1} and 21 lb P_2O_5 fertilizer acre^{-1} to each observation and each future crop in the data set. The *optimal uniform* simulation chooses fertilizer rates for each future crop that maximize the discounted expected gross margin (wheat sales less fertilizer cost) for all 7 future crops, using data set mean soil fertility measures. The recommended rates of N and P fertilizer (one N and one P level for each of 7 future crops) are assigned to each observation in the data set and model simulated results summarized. The *optimal VRT* simulation chooses the fertilizer rates for each future crop which maximize the discounted expected gross margin for all 7 future crops, using soil fertility measures unique to each observation in the data set. A dollar advantage in the right column is a total 7- or 5-crop advantage, discounted to the present.

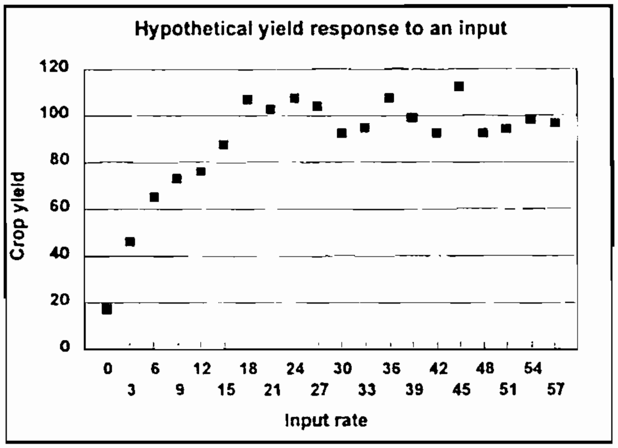
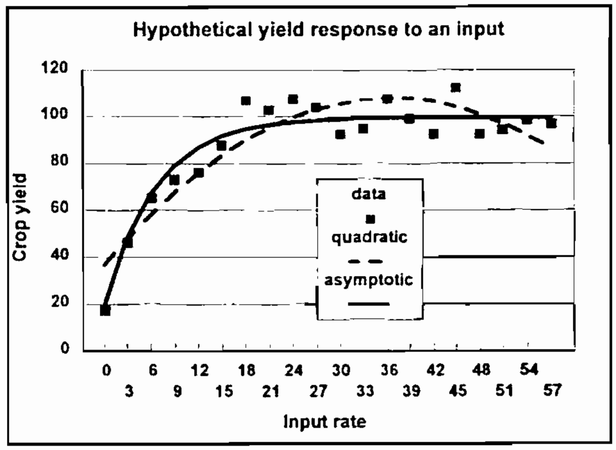


Figure 1



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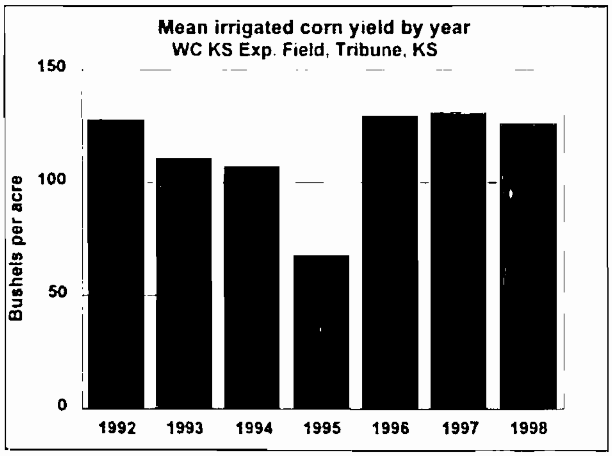

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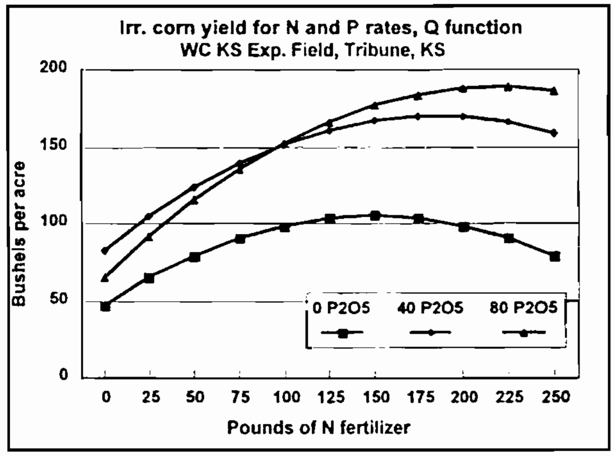

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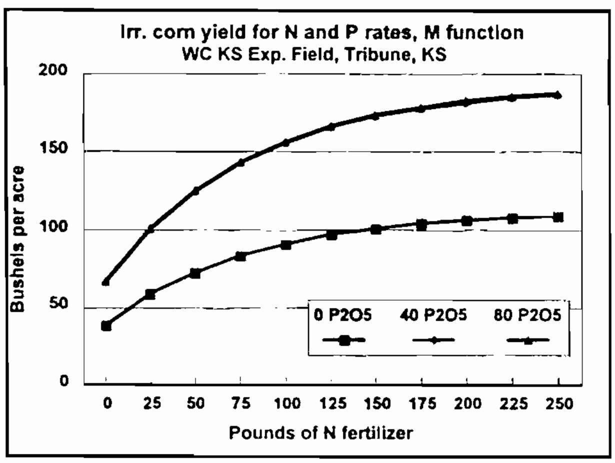

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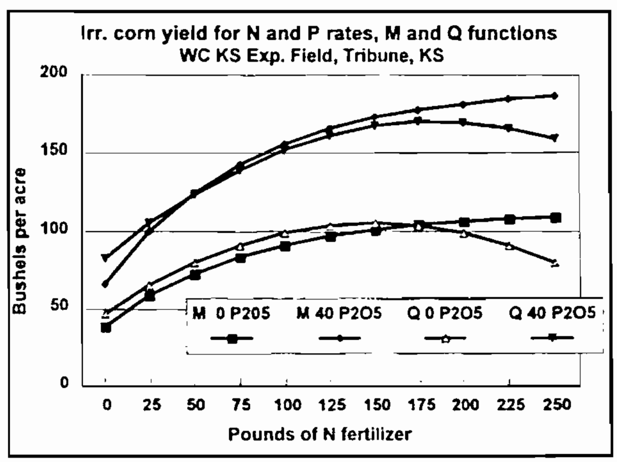

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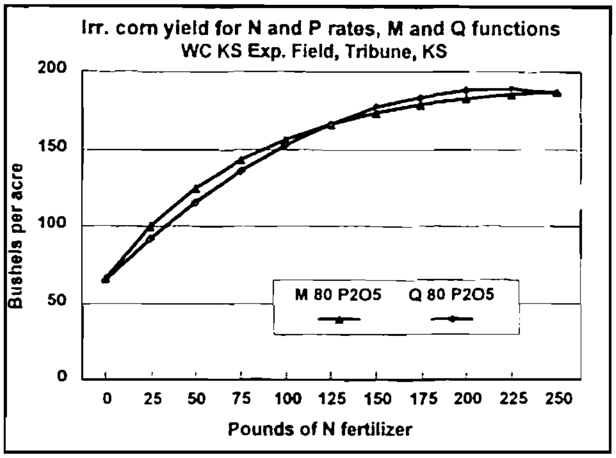

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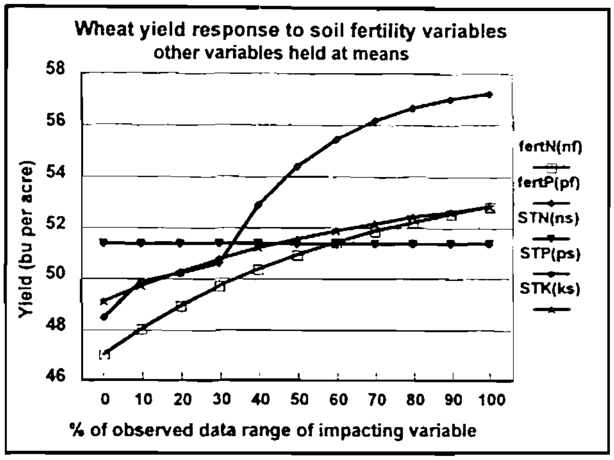

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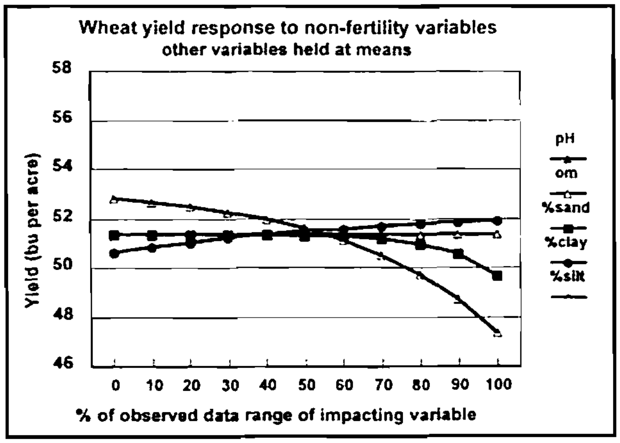

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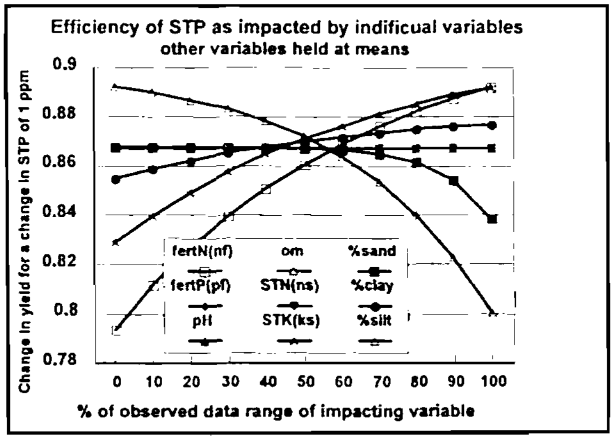

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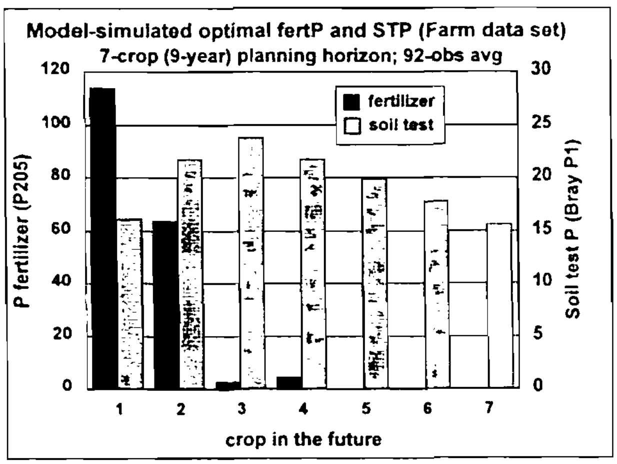

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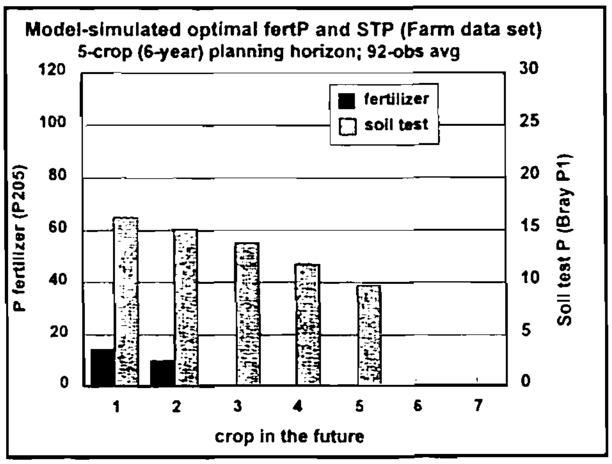

Figure 12

PROCEEDINGS OF THE

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Program Chair:
Dr. Ed Lentz
Ohio State University Extension
952 Lima Avenue
Findley, OH 45840
419/422-6106

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772 – 22nd Avenue South
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